Supplemental Notes on Wave Equation
Numerical Solutions

We shall examine the numerical solution to the wave equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

Here $u(x,t)$ is a scalar quantity propagating with speed a, a real constant which may be positive or negative. For our discussion here let $a = 1$.

We apply the first order Euler explicit time differencing and examine using 2nd order central differencing for $\frac{\partial u}{\partial x}$.

Assume a mesh domain $(j\Delta x)$, centered about $j = J$ with points $J-3, J-2, J-1, J+1, J+2, J+3$ on either side of J. The notation in space and time is

$$u_j^{(n)} = u(j\Delta x, n\Delta t)$$

We initialize to solution with I.C. $(n = 0)$

$$\begin{aligned} u_j^{(0)} &= 0\,; \quad j < J & (2)\\ u_j^{(0)} &= 1\,; \quad j = J & (3)\\ u_j^{(0)} &= 0\,; \quad j > J & (4) \end{aligned}$$

The time advance and space differencing algorithm is

$$u_j^{(n+1)} = u_j^{(n)} - \frac{\Delta t}{2\Delta x}(u_{j+1}^{(n)} - u_{j-1}^{(n)}) \tag{5}$$

Now lets examine the solution as we advance one time step to $n = 1$.

$$\begin{aligned} j < J-1 \quad &\rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{2\Delta x}(0-0) = 0 & (6)\\ j = J-1 \quad &\rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{2\Delta x}(1-0) = -\frac{\Delta t}{2\Delta x} & (7)\\ j = J \quad &\rightarrow \quad u_j^{(1)} = 1 - \frac{\Delta t}{2\Delta x}(0-0) = 1 & (8)\\ j = J+1 \quad &\rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{2\Delta x}(0-1) = \frac{\Delta t}{2\Delta x} & (9)\\ j > J+1 \quad &\rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{2\Delta x}(0-0) = 0 & (10) \end{aligned}$$

After one time step (Δt) we should have had just a shift of the I.C. over by Δx. For example, if $\frac{\Delta t}{\Delta x} = 1$ then we move one grid point per time step. Figure 1 shows the error after one time step. If we take another step we obtain Figure 2, and after 3 time steps, Figure 3.

The process of spreading and growing the solution continues and eventually the solution diverges to infinity.

Now in contrast, what if we had used a 1st order one sided difference instead of the 2nd order central difference.

The time advance and space differencing algorithm is

$$u_j^{(n+1)} = u_j^{(n)} - \frac{\Delta t}{\Delta x}(u_j^{(n)} - u_{j-1}^{(n)}) \tag{11}$$

Now lets examine the solution as we advance one time step to $n = 1$.

$$j < J-1 \quad \rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{\Delta x}(0-0) = 0 \tag{12}$$

$$j = J-1 \quad \rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{\Delta x}(0-0) = 0 \tag{13}$$

$$j = J \quad \rightarrow \quad u_j^{(1)} = 1 - \frac{\Delta t}{\Delta x}(1-0) = 1 - \frac{\Delta t}{\Delta x} \tag{14}$$

$$j = J+1 \quad \rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{\Delta x}(0-1) = \frac{\Delta t}{\Delta x} \tag{15}$$

$$j > J+1 \quad \rightarrow \quad u_j^{(1)} = 0 - \frac{\Delta t}{\Delta x}(0-0) = 0 \tag{16}$$

Here we have the solution moving in an exact manner only if $\frac{\Delta t}{\Delta x} = 1$. In the case $\frac{\Delta t}{\Delta x} < 1$ the numerical solution does move to the right, but also spreads and decays. The initial condition is shown in Figure 4 and one can see the error generated in just one time step in Figure 5 and after 3 time steps in Figure 6

At least this appears to be stable in the sense that the solution goes to zero and not infinity. If on the other hand, we use $\frac{\Delta t}{\Delta x} > 1.0$, we again get instability as shown after one time step in Figure 7 and after 3 time steps in Figure 8.

Note that stability and accuracy of numerical solution depends both on Δx and Δt, in fact, $\sim \frac{\Delta t}{\Delta x}$

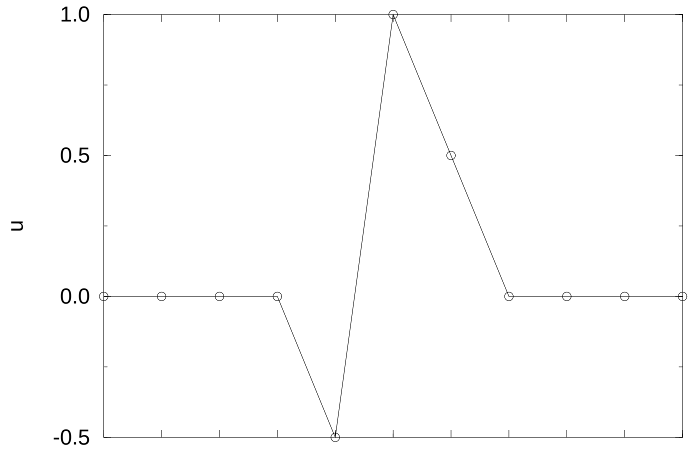

Figure 1: One time step of Eq. 5 for $\frac{\Delta t}{\Delta x} = 1.0$.

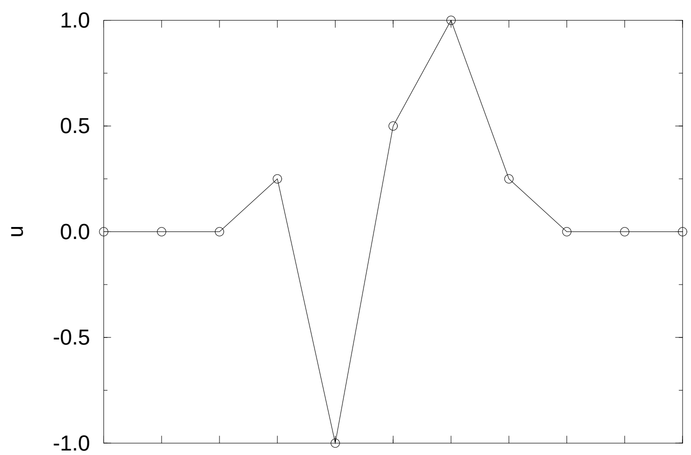

Figure 2: 2 time steps of Eq. 5 for $\frac{\Delta t}{\Delta x} = 1.0$.

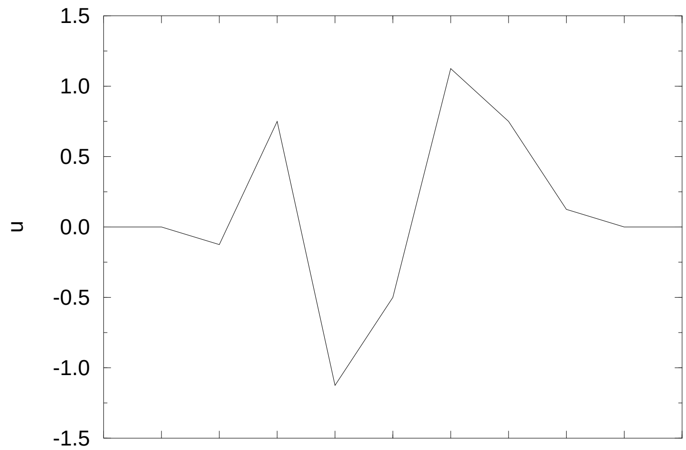

Figure 3: 3 time steps of Eq. 5 for $\frac{\Delta t}{\Delta x} = 1.0$.

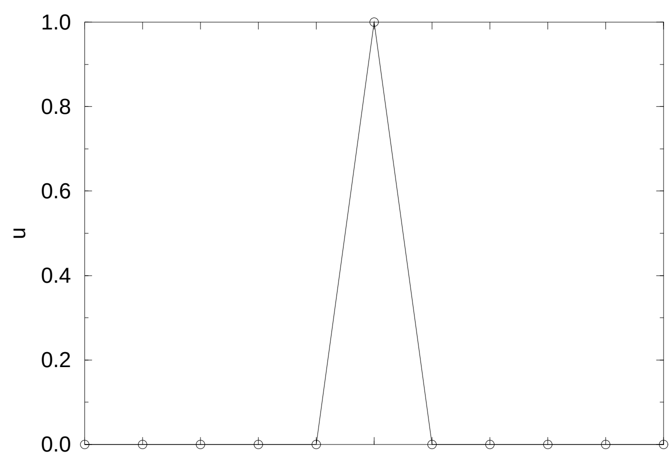

Figure 4: Initial Condition .

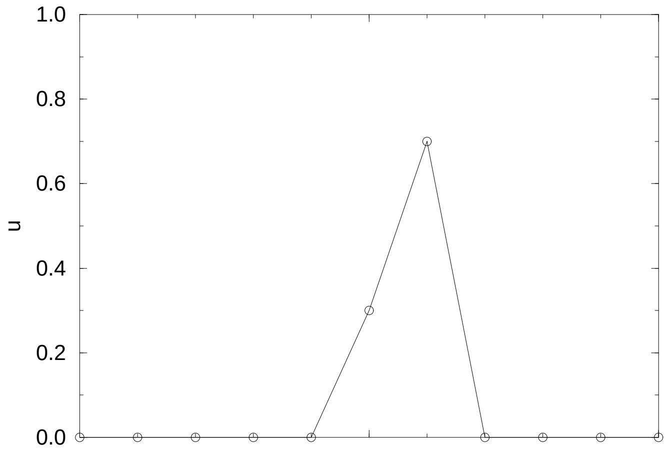

Figure 5: 1 time steps of Eq. 11 for $\frac{\Delta t}{\Delta x} = 0.7$.

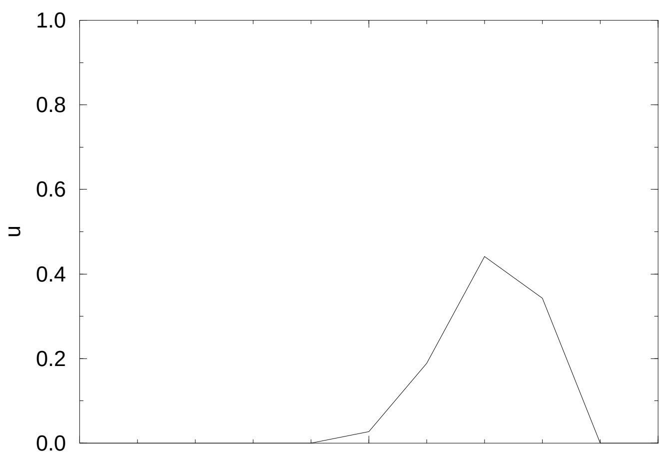

Figure 6: 3 time steps of Eq. 11 for $\frac{\Delta t}{\Delta x} = 0.7$.

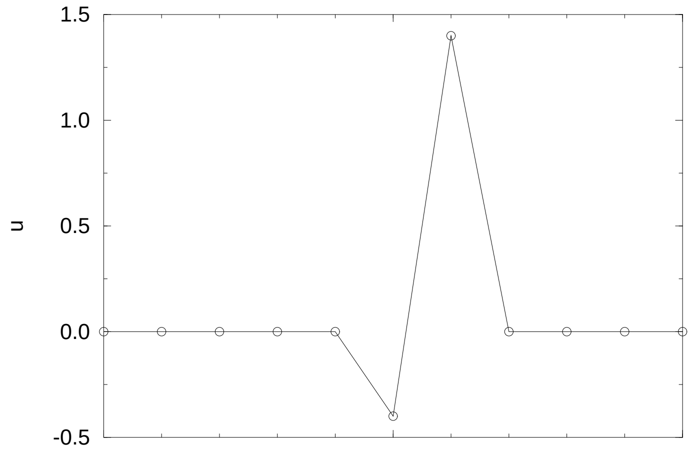

Figure 7: 1 time steps of Eq. 11 for $\frac{\Delta t}{\Delta x} = 1.4$.

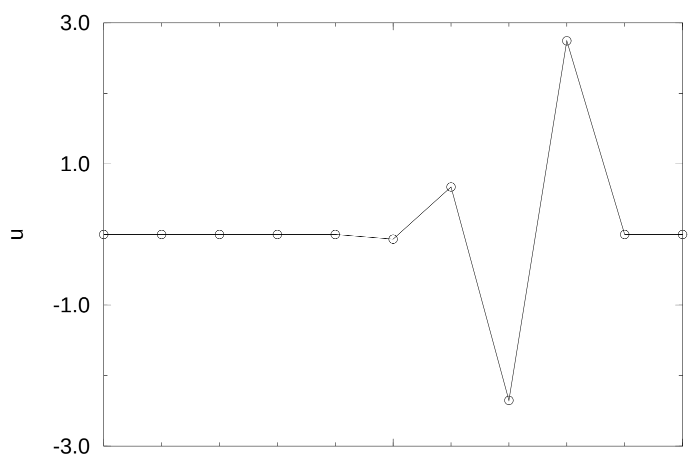

Figure 8: 3 time steps of Eq. 11 for $\frac{\Delta t}{\Delta x} = 1.4$.